STUDENTS' CONCEPTIONS OF DERIVATIVE GIVEN DIFFERENT REPRESENTATIONS

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In Calculus, students are often both presented problems with and taught to use three interconnected types of representations: symbolic, graphic, and numeric. However, students often fail to notice the relationship between mathematical objects (and even the same object) that are presented using different types of representations. Using the APOS framework and student interviews, this study explores ways in which students conceive of tasks involving derivatives that are posed using the different representational types. Patterns are drawn among: problem conception and representation type; problem conception and task type; schema level and representation type; and schema level and problem conception. Though there were patterns between representation and student conception, stronger patterns emerged between task type and student conception.

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Introduction

In calculus, the concept of derivative is often taught using three interconnected types of representations: symbolic, graphic, and numeric (Schwarz & Hershkowitz, 2001). However, students often do not recognize and use the relationships between them well (Asiala et al., 1997; Aspinwall, Shaw, & Presmeg, 1997; Schwarz & Hershkowitz, 2001). This may be for a variety of reasons, such as that algebraic representations are typically used more on tests (Aspinwall et. al, 1997), that instruction focuses on specific translation skills instead of a discussion of relationships between representations (Schwarz & Hershkowitz, 2001), and that students are often asked to translate in one direction, but not in reverse (Asiala et al., 1997).

Anna Sfard (1991) discussed the interaction between visual (graphs and pictures) and analytic (equations and numbers) representations and grouped students' conceptions into two complementary categories: operational and structural. She claimed that analytic notation encourages an operational conception, for which the focus is the procedure, whereas visual notation encourages a structural conception of the mathematical entity as an object. Therefore, students might tend to think about derivative differently depending on the representation presented in a problem. For example, if a student is presented with a graph of a function, he may tend to use a structural conception, but if presented with an equation he may hold an operational conception. This does not give information about the student who is presented with a tabular representation, or that needs to translate between representations in order to solve a problem.

Similar to Sfard's operational and structural conceptions, Dubinsky posits the APOS theory, with which student understanding of a mathematical entity falls into one of four conceptions: action, process, object, and schema (e.g. Dubinsky & McDonald, 2001; Maharaj, 2013). The action and process conceptions align with Sfard's operational conception, while the APOS object conception aligns with Sfard's structural conception (Maharajh, Brajlall, & Govender, 2010; Tall, 2013). However, while a student may possess several different conceptions of a mathematical entity, the student may not access all of these equally in all situations (Baker, Cooley, & Trigueros, 2000), particularly when comparing visual and analytic representations, both during instruction and problem solving (Maharajh et. al, 2010).

Dubinsky & McDonald (2001) and Baker et. al (2000) also discuss the notion of the *triad*, a fixed order of three stages of schema development (intra, inter, and trans) in which students progressively

see more connections between events or objects. Students begin at the *intra* stage, where they see mathematical entities as disconnected objects, and as they develop connections between objects, they progressively advance through the *inter* stage to the *trans* stage, in which they possess a coherent schema with which they can reason about one or more related objects in a meaningful way. One might expect this triad to exhibit a correspondence of stages to the APOS conceptions of cognition, with the action conception corresponding to the intra stage, the process conception corresponding to the inter stage, and the object conception corresponding to the trans stage. This observation brings into focus the following research question: how are students' conceptions of derivative related to the ways in which they make connections within and among different representations? For example, is a student who is operating at the intra stage unable to utilize an object conception for a mathematical entity because that would require connections that are not yet made? Or, if a student displays an object conception of derivative when presented a graph, but an action conception when presented a table, would the student be able to draw the necessary connections between the information given by the graph and the table?

Framework

APOS theory has been used by a variety of researchers to analyze students' procedural and conceptual knowledge (e.g. Asiala et. al, 1997; Baker et. al, 2000; Hähkiöniemi, 2006; Maharaj, 2013). APOS stands for *action*, *process*, *object*, and *schema*, which are categories of conceptions that an individual might hold of a mathematical entity (Maharaj, 2013). Understanding APOS conceptions allows us to form a *genetic decomposition*, or model of the specific conceptions that a learner might make (Asiala et. al, 1997) to better describe students' understanding of mathematical concepts (Dubinsky & McDonald, 2001; Maharajh et. al, 2010).

An individual who has an action conception is able to carry out a step by step set of instructions or a written algorithm, but is able to do little more (Asiala et. al, 1997; Dubinsky & McDonald, 2001) or to control the action (Hähkiöniemi, 2006). While an action is always in response to a stimulus that is perceived as external, a process performs the same operation as an action, but may be contained completely in the mind of the individual (Maharaj, 2013). This internalization allows the individual to be able to reverse or compose operations (Dubinsky & McDonald, 2001). An object is formed as a complete entity in the individual's mind when a process can be transformed and acted on as a totality (Dubinsky & McDonald, 2001; Hähkiöniemi, 2006; Maharaj, 2013). A schema is a collection of actions, processes, and objects coordinated by the individual (Baker et. al, 2000; Dubinsky & McDonald, 2001).

Because an individual's schema is an evolving composition of actions, processes, and objects, it is more of a meta-cognitive development for problem solving than a conception in itself (Baker et. al, 2000; Dubinsky & McDonald, 2001). This means that at any given moment, when an individual uses either an action, process, or object conception to solve a particular problem, the individual is concurrently operating within their schema as well. For this reason, this study describes only the APOS conceptions of action, process, and object, and assumes that the individual is always operating within some personal schema.

Table 1 gives examples of what might be observed of a student using each conception if a problem involving derivative is given using each of the three main representations: table, equation, and graph. It is important to note that even though these conceptions seem to evolve from one to the next, the development of action, process, and object is not necessarily in that order (Dubinsky & McDonald, 2001). Though this study explores the student's conceptions as a snapshot, and not being developed over a substantial period of time, this is still a valuable reminder for the current study because, for example, it is possible for an individual to use an object conception without having ever thought through the action or process involved first.

It is a purpose of this study to determine if a student will tend to use different APOS conceptions given different problem representations. For example, if given a table, a student can perform steps to approximate a derivative at a point only when prompted for the steps, the student would be at the action conception. However, if the student is given the same question with a graphical representation and has no difficulty recalling that the derivative at a point being the slope of the line tangent to the graph, drawing the line, and approximating the slope, the student would be at categorized as having a process conception. This genetic decomposition is used in the analysis and discussion of students' responses to interview questions in this study.

Table 1: APOS Conceptions of Derivative Given Different Representations.

Conception	Table	Equation	Graph
Action	The student follows steps when prompted to find slope and write equation of tangent line.	The student can use steps to find equation for derivative and tangent line.	The student can use steps when prompted to find slope from graph and draw the tangent line.
Process	The student can easily apply procedures using tabular data, and can catch errors and explain reasoning.	The student can easily manipulate equations dealing with derivatives and provide some rationale.	The student can notice and use parts of the graph to answer questions about tangent line and derivative.
Object	The student uses the table to reason about derivative and solve problems.	The student uses the equation to reason about derivative and solve problems.	The student uses the graph to reason about derivative and solve problems.

APOS theory has been further refined through the introduction of the *triad* to recognize the evolving nature of schemas (Baker et. al, 2000). The triad consists of three stages, *intra*, *inter*, and *trans*, which occur in that order (Baker, Cooley, & Trigueros, 2000). As is suggested by their names, at the intra stage, the individual is concerned with and isolates the single action, process, or object of focus, while at the inter stage, relationships are formed between cognitive entities (Dubinsky & McDonald, 2001). For example, a student given a graph at the intra stage may be able to draw a tangent line but not look for numerical information to find its slope. The student at the inter stage would use the graph to find numerical values or an equation that could then be used to solve the problem, but these connections would still be isolated and superficial and while the student may reason about why a particular representation is chosen, the student does not reason about more than one representation at a time. An individual operating at the trans stage exhibits a schema with the most coherence, for example, recognizing all situations requiring the computation of a derivative at a point as interconnected (Baker et. al, 2000). Table 2 describes what might be observed of a student at each stage given different representations.

Methods

Four students were interviewed using a series of calculus questions to determine their understanding of derivatives. The researcher recruited volunteers from an AP Calculus class at a mid-sized, southern high school. The volunteers from this class are considered to be typical students who take AP Calculus at this school. They have all taken two courses in algebra, a course in geometry, and a preparatory class for calculus (Pre-calculus). Two are female and two are male, with one of the females in twelfth grade and the other three students in eleventh grade. All four of the students interviewed perform well in their class, with an average grade of either an A or high B. At

Table 2: Triad Stages for Derivative Given Different Representations.

Triad stage	Table	Equation	Graph
Intra	The student makes decisions using only a tabular representation.	The student makes decisions using only equations.	The student makes decisions using only a graphic representation.
Inter	The student uses the values in the table to construct a graph or equation.	The student uses the equation to find numerical values or construct a graph.	The student uses the graph to find numerical values or an equation.
Trans	The student reasons about connections between the table and an equation and/or graph to solve the problem.	The student reasons about connections between the equation and table and/or graph to solve the problem.	The student reasons about connections between the graph and the equation and/or numerical values to solve the problem.

the time of the interviews, students had been in the calculus class for about eight months, and were studying integrals. They began their study of derivatives several months before the interviews took place, completed a unit focused on derivatives, and have continually used and applied them to other calculus problems since.

Each interview was conducted outside of school hours at a local public library, and lasted between 30 and 45 minutes. Interviews were recorded using a Livescribe pen and paper, which records the interview and connects the audio recording to the written work.

Each of the three interview questions presented the student with information about a problem using a different representation (table, equation, or graph). Question 1 (tabular representation) is shown in Figure 1 as a representative of each of the questions asked. Though the representation is different across the three questions, the tasks in each part are the same across questions. The reason for the repetition of the four tasks across interview questions is to determine if students tend to use a particular conception over another for a task when given different representations. In addition, these tasks were designed to assess the student's ability to translate between representations, thus assessing the triad stage that the student uses.

#1: The following table gives values of a differentiable function f(x) for different values of x:

X	-1	0	1	2	3	4	5	6
f(x)	-2	2	5	8	12	8	4	7

a. Approximate f'(1).

b. Approximate $\lim_{h\to 0}\left(\dfrac{f(4+h)-8}{h}\right)$

c. If f'(-1) = 5, write the equation of the line tangent to f(x) at x = -1.

d. Is f'(x) probably increasing or decreasing at x = 3?

Figure 1: Interview Question #1

The data from each interview was then analyzed to determine each student's APOS conception and triad stage for each interview question. The results were then reviewed to determine if there is a

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correlation between the representation given to the student, the APOS conception that the student used, and the triad stage at which the student operated.

Results

While there was at least one instance for each APOS conception given each representation, some strong patterns emerged. Table 3 compares the given representation to the APOS conception that the student used. Since each problem contained four tasks, and each was asked to four students, the total for each row is sixteen. The *no determination* column was added because there were some problem parts where the student either said that they did not know how to do it or else created an incorrect "rule" with no justification.

Table 3: Students APOS Conceptions Across Given Representations.

Representation	No determination	Action	Process	Object
1. Table (numeric)	7	2	3	4
2. Equation (analytic)	1	3	9	3
3. Graph (visual)	5	1	5	5

Students felt the most confident when given the equation, with which they almost all immediately started writing out answers, compared to where they generally stopped to think about a course of action given the other representations. They exhibited a strong process conception in that there was a routine and they were following it, but they also knew why they followed each step and could give appropriate justifications for each part of their answers.

A stronger pattern that emerged was that between the type of information asked for in each task and the students' APOS conception used in approaching the task. Table 4 compares the task type to the APOS conception used.

Table 4: Students APOS Conceptions Across Task Type.

Task Type	No determination	Action	Process	Object
a. Find derivative (at a point)	3	1	5	3
b. Find derivative by limit.	9	3	-	-
c. Write equation of tangent line.	-	1	10	1
d. Is the derivative increasing or decreasing?	2	-	2	8

Students had the most difficult time with the limit part of each question. All four students verbally said that they had forgotten how to find limits, but this again was dependent on the given representation. In the questions that give the table and graph, students could describe what a limit is, but when actually approaching the problem, mostly said that they do not "remember how to do it." Given the equation representation, three students of the four made much more of an attempt to try different algebraic rules until they came to an answer for the limit. The student in the following excerpt demonstrates an action conception in trying to find the limit of a difference quotient given an equation (A denotes the student, R the researcher):

A: [draws graph] I feel like this is not the way to do it, but if you plug in -1… you get 33, and at 1… it's 51. [sighs] I know there is another way to do this, it's been so long.

[The student skipped to parts c and d and then came back to b.]

R: So when you were doing limits, did you just plug numbers in, or did you cancel first and then plug numbers in? Did you do anything with simplifying?

A: Oh, wait, wait, ok. Ok, um, you factor, not factor, but, um… 3 times 2, 6, plug 3 times h equals 3h [continues to simplify part b at the right of Figure 1]… Yes! Ok, sorry, yeah, so

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then you factor h out, you have 9h plus 42 over h, those cancel, and nine times 0 plus 42, so the limit at that point is 42!

In contrast, the students held a process conception in ten of the twelve instances of writing the equation of a tangent line, and did this part of each question with the least effort, regardless of representation, and also regardless of whether they could find the derivative. The student who struggled with the limit went through the solution to using the equation of the function to find the equation of the tangent line at x = 0 almost without pause:

A: Ok, so all you gotta do is plug in 0 to the derivative so that's 6, so that's the slope of your tangent line. And to find y, you plug it into your original equation, that's 1, so I need slope-intercept form, b = 1, the equation of the tangent line is 6x + 1 at x = 0.

The students used an object conception in eight of the twelve instances when thinking about how the derivative changes, regardless of given representation. In this excerpt, the student is reasoning about if the derivative is increasing or decreasing given the graph of the function:

B: All right, k' is the rate of change of the graph itself, and it's decreasing the entire time.
R: What's decreasing?
B: Well, the original function is decreasing all throughout this set, but it's getting less and less steep of a decrease, which means that the, uh, the derivative or k' is increasing on this period because it is uh, it's sort of like, it's approaching 0 which it does, it hits at 0 or somewhere near there it looks like, and it's negative before that because it's going downwards, and it's positive afterwards because it's going up afterwards. So in between here it's negative, or increasing, because it's approaching zero from a negative number.

However, the way that students thought about finding the derivative to begin with depended more on the representation. All of the students used the chain rule quickly and easily and could give appropriate justifications when given the equation. In contrast, there is only one instance of a student that was given a graph or table who held a process conception of finding a derivative. For these representations, the students were much more likely to reason about the function as an object, or to decide that they did not know how to get the answer. Students also looked at the graph and table in much the same way. One student actually remarked when they got to #3 (the tasks using the graphical representations) that they looked a lot like #1 (the tasks using the tabular representation), even though the questions were identical for all three representations:

C: Ok, so this is pretty similar to the first one.
R: Ok, in what way?
C: Well, I mean it gives you data, except it's in a graph, not a table. And it uh, in general it's pretty much the same thing. When it says approximate k'…

The given representation did not seem to make much of a difference in the schema level that the students employed (see Table 5). The intra schema, for which students stay within the given representation, seems to be more common with equations; however, that is affected by the students being willing to try the limit question when given the equation, effectively moving three scores from

Table 5: Schema Stage Across Representation

	No determination	**Intra**	**Inter**	**Trans**
1. Table (numeric)	4	6	3	3
2. Equation (analytic)	1	8	4	3
3. Graph (visual)	4	4	6	2

no schema to intra. The inter and trans schemas show similar amounts of instances across representations.

There was a very strong relationship between APOS conception and the schema employed. Table 6 shows the amount of instances of reasoning within each conception at each schema level.

Table 6: Comparison of APOS Conception and Triad Stage

	No determination	Intra	Inter	Trans
No determination	9	3	1	-
Action conception	-	3	3	-
Process conception	-	10	7	-
Object conception	-	2	3	7

Not surprisingly, most instances in which the conception is not determined also had no determined schema with which to deal with the problem. This occurred most often with the limit task, but also in two instances of finding the derivative at a point and in determining whether the derivative was increasing or decreasing. At the intra and inter level, students with undetermined conception substituted in numbers or came up with some false "rule," which they applied to the problem.

All of the students interviewed revealed an action or process conception at least part of the time, and in all instances that the action or process conception was held, the student operated at the intra or inter schema level. This means that the student either stayed entirely within the given representation, or that they translated part of the information given to a different representation without reasoning about the relationship between representations. For example, when asked to find a derivative given an equation, all four students operated under a process conception and simply applied the chain rule. They stayed within the given representation, and when prompted, they explained how the chain rule works, but did not include other representations in their description. When asked to find the equation of the tangent line at a point given an equation, all four again used a process conception, and one student stayed in the intra schema level, but the other three moved into an inter level, translating between representations to either complete the task or describe it, but not reasoning about the connections between representations.

Finally, students who used an object conception did use all three schema levels, but tended to be at the trans level more often, reasoning about connections between representations in order to complete the task. For example, in reasoning about whether the derivative is increasing or decreasing a student set the derivative equal to zero and solved. She then explained that this would help her find whether there is a local maximum or minimum. She sketched graphs of a minimum and maximum and gestured the slope along each graph to reason about which derivative would be increasing or decreasing. The student then went on to sketch a number line and populate the number line with the solution she found by setting the derivative to zero. She related the number line to her graphs by cross-comparing the points she had made before determining that the derivative is going from negative to positive so it is increasing. Though it seems like she answered the question several times in different ways, in her mind she only answered it once because her answer consisted of connecting all of the pieces of her schema in forming her object conception of the derivative to complete the task.

Discussion

This study uses APOS theory to guide the theoretical framework for analyzing students' conceptions of derivatives in calculus. Four students participated in task-based interviews in which tasks were presented using three different representations. While findings agree with the existing literature that students tend to use a process conception more when given an equation and less when

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given a graph, there is also evidence to suggest that a relationship may exist between task type and student conception. In particular, students tended to use an action or process conception when determining the derivative of a function or the equation of the tangent line, and an object conception when reasoning about the change in derivative. The students in this study also used an intra or inter schema in conjunction with their action or process conception, and only operated on the trans schema level when using an object conception. However, even with the object conception, the trans schema level was only employed in seven out of 12 instances.

This study contributes to our understanding of students' conceptions of derivative, which can be useful for instructors in planning course instruction and assessment. For example, teachers may find it helpful to be cognizant of the conceptions that students tend to have when given different tasks such as writing the equation of a tangent line or reasoning about the change in derivative for a function. It may also be useful to consider students' schematic levels for thinking of derivative when considering problems to pose that may lead to action, process, or object conceptions by students.

Students complete many other tasks in calculus that were not analyzed in this study, and further research is needed to determine if the relationship between task type and APOS conception relies on cognitive demand, type of representation, class routine, or other contributing factors. Further research is also needed to determine the strength of the relationships between representation and conception, between task type and conception, and between conception and schema level for students in a variety of calculus classes.

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